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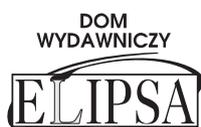
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THE VALUE OF A PREPAYMENT OPTION IN A FIXED RATE MORTGAGE: INSIGHTS FROM BREAKEVEN VOLATILITY

1. INTRODUCTION

A fixed rate loan – i.e. a contract wherein the interest rate paid by the client is fixed throughout the duration of the contract – carries three main sources of risk for the originating bank. The first one, as in any other loan, there is the credit risk related to the default of the borrower. The second one, there is interest rate risk, namely the risk that market rates increase and exceed the rate at which the contract was concluded. Finally, there is a prepayment, or callability, risk related to the fact that borrowers may decide to pay their loan back prior to its maturity (i.e. “prepay”). If originators hedge the interest rate risk of their mortgage portfolios with simple interest rate swaps, then whenever such prepayment occurs, they have to unwind some or all of the hedging positions which – given that prepayment tends to occur at lower interest rates – results in losses. The cost of a fixed rate mortgage – the interest rate being agreed in the contract – should compensate the originator for bearing these three sources of risk. Therefore, the rate on a fixed rate mortgage can be decomposed into three elements: (i) fixed-for-floating interest rate swap rate with maturity corresponding to the maturity of the loan; (ii) credit spread; (iii) pre-payment spread. Out of these three, this is the prepayment spread that is most difficult to estimate. After all, the borrower’s option to call the loan at face

* Juliusz Jabłecki works at University of Warsaw and National Bank of Poland; no part of this article should be taken to represent the official view of the NBP.

value is essentially American in nature, i.e. it can be exercised at any time prior to maturity. Hence, estimating the fair value of a prepayment option requires not only a pricing model to handle the early exercise feature, but also a rich enough universe of plain vanilla calibration instruments – in this case ideally co-terminal European interest rate swap options, i.e. swaptions¹. A basic requirement for any model used for valuing exotic derivatives – such as options with early exercise features – is that it prices exotics consistently with their simpler counterparts quoted on the market. This ensures that the price of an illiquid exotic product is “at par” with prices of plain vanilla liquid products often used to hedge or replicate it. In the absence of such a liquid market in basic interest rate derivatives, estimates of prepayment option value can be biased and the resulting prepayment spreads distorted. Thus, underdevelopment of an interest rate derivatives market can be a hindrance for the fixed rate mortgages and other products with callability features.

This paper tries to contribute to the vast literature on managing prepayment risk² by proposing a methodology for estimating the value of a prepayment option in the absence of a deep and liquid market in interest rate swaptions. In such circumstances there is no implied volatility surface of plain vanilla European swaptions with which the more exotic early-exercise pricing model can be made to agree, which compounds the uncertainty surrounding the valuation of American-style payoffs. The proposed approach builds on the concept of breakeven volatility Dupire³, i.e. the volatility level at which the price of the option on a historical date may be replicated by the P&L from continuously delta hedging it until expiry. Although Dupire originally proposed the concept for commodities and currencies with illiquid or non-existent options markets⁴, we show that it can be readily applied to options on interest rate underlyings, and in particular swaptions. Such breakeven volatilities can be calculated for different swaption maturities, strike rates and underlying swap tenors yielding a full co-terminal swaption volatility surface conditioned on the realized historical zero coupon bond prices and swap rates. By construction, the resulting implied volatilities will be backward-looking. However,

¹ This is the so called option-theoretic or endogenous approach to the estimation of prepayment risk, see e.g. Davidson A., Levin A., *Mortgage Valuation Models: Embedded Options, Risk, and Uncertainty*, Oxford University Press 2014 or Qu D., *Manufacturing and managing customer-driven derivatives*, John Wiley & Sons, Chichester 2016, West Sussex, United Kingdom chap. 19 for a comprehensive discussion and alternative perspectives.

² See e.g. Kau J.B., Keenan D.C., *An overview of the option-theoretic pricing of mortgages*, Journal of Housing Research 1995, 6(2), 217; Collin-Dufresne P., Harding J.P., *A closed form formula for valuing mortgages*, The Journal of Real Estate Finance and Economics 1999, 19(2), 133–146; Agarwal S., Driscoll J.C., Laibson D.I., *Optimal Mortgage Refinancing: A Closed-Form Solution*, Journal of Money, Credit and Banking 2013, 45(4), 591–622.

³ B. Dupire, *Pricing with a smile*, Risk 1994, 7(1), 18–20.

⁴ Dupire’s breakeven volatility approach has been implemented in the widely-used Bloomberg system e.g. for Nigerian Naira and Kenyan Shilling.

they can serve as a rough guide for where volatility levels should be given historical data. Applying this method to the Polish historical interest rate curve, we find that the implied (breakeven) volatility surface exhibits a pronounced dependence both on strike and swaption term/tenor, i.e. so called smile and term structure. The dependence of swaption implied volatilities on strike is a well documented phenomenon in markets where swaption quotes are available. However, it is also inconsistent with the Black-Scholes valuation framework as it suggests that some swaptions are priced as if the same underlying swap rate moved by 4 bp a day and some – 8 bp a day, which is nonsense. To accommodate market patterns while retaining the completeness and simplicity of the Black-Scholes framework we propose a local volatility model in which the swap rate volatility is made time and state dependent, consistently with the breakeven volatility surface. Concretely, building on Gatarek and Jablecki⁵ we derive an equation for the unique state-dependent diffusion coefficient consistent with breakeven swaption volatilities, linking it to the dynamics of the entire interest rate curve. We then use the diffusion to price the prepayment option, *qua* a Bermudan receiver swaption implicitly contained in a fixed rate mortgage contract using data from the Polish market as of January 2017. The mortgage spread component related to the prepayment option price proves to be quite significant, stressing the importance of an adequate risk management of the inherent callability feature and possibly explains why fixed rate mortgage products have so far struggled to develop in Poland.

2. NOTATION AND DEFINITIONS

2.1. Financial market instruments

We start by defining the main instruments and a notation we are going to work with throughout. At this point, our approach is an independent model, but we assume an interest rate model of the Heath-Jarrow-Morton type to facilitate the presentation. Concretely, let $P(t, T)$ be time t price of a zero coupon bond maturing at time T such that $P(t, t) = 1$ for every t . We assume there exists a frictionless and arbitrage-free market for zero coupon bonds such that $P(t, T)$ exists for every $0 < t < T < \infty$ and for a given t , $P(t, T)$ is differentiable with respect to maturity time T . The instantaneous forward rate $f(t, T)$ with maturity T contracted at t is defined by

$$f(t, T) \equiv -\frac{\partial \ln P(t, T)}{\partial T} \iff P(t, T) = \exp\left(-\int_t^T f(t, s) ds\right). \quad (\text{Eq. 2.1})$$

⁵ D. Gatarek, J. Jablecki, *A local volatility model for swaptions smile*, Journal of Computational Finance 2016, Forthcoming.

The instantaneous spot rate $r(t)$ – i.e. the short rate – is defined by the condition

$$r(t) \equiv f(t, t) \quad (\text{Eq. 2.2})$$

and can be interpreted as capturing the locally risk-free return from a continuously compounded money market account $B(t) \equiv \exp \left\{ \int_0^t r(s) ds \right\}$. The short rate is not to be confused with a continuously compounded spot interest rate, $R(t, T)$, defined as

$$R(t, T) \equiv - \frac{\ln P(t, T)}{\delta(t, T)}, \quad (\text{Eq. 2.3})$$

where δ , year fraction, stands for the chosen time measure between t and T . Finally, we also introduce simply compounded spot interest rate, referred to as LIBOR rate $L(t, T)$:

$$L(t, T) \equiv \frac{1 - P(t, T)}{\delta(t, T)P(t, T)}, \quad (\text{Eq. 2.4})$$

along with a time t forward rate between two dates T and S :

$$L(t; T, S) \equiv \frac{P(t, T) - P(t, S)}{\delta(T, S)P(t, S)}. \quad (\text{Eq. 2.5})$$

Define now a uniformly spaced tenor structure:

$$0 = T_0 < T_1 < \dots < T_M \quad (\text{Eq. 2.6})$$

and set $\delta_n = T_n - T_{n-1}$ for $n = 1, \dots, M$. A **fixed-for-floating interest rate swap** (IRS) with unit notional, fixed rate (coupon) K , and a specified tenor structure $\mathcal{T} = \{T_n\}_{n=\alpha+1}^\beta$ is a contract whereby two parties exchange differently indexed cash flows over a pre-agreed time span. Specifically, on each date $T_n \in \mathcal{T}$, the fixed leg pays $\delta_n K$, whereas the floating leg pays the floating LIBOR rate given by the formula:

$$\frac{1 - P(T_{n-1}, T_n)}{\delta_n P(T_{n-1}, T_n)} \delta_n. \quad (\text{Eq. 2.7})$$

When the fixed leg is paid, the IRS is called a “payer,” conversely the swap is called a “receiver.” The forward swap rate $S_{\alpha, \beta}(t)$ corresponding to the tenor structure \mathcal{T}

is the rate in the fixed leg that sets it equal to the floating leg and hence makes the net present value of the transaction equal zero:

$$S_{\alpha,\beta}(t) \equiv \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{n=\alpha+1}^{\beta} P(t, T_n) \delta_n}. \quad (\text{Eq. 2.8})$$

When setting $\alpha = 0$, it can be immediately noticed that the spot swap rate for a contract maturing at T_β reduces to $(1 - P(0, T_\beta)) / \sum_{n=1}^{\beta} P(0, T_n) \delta_n$.

A European payer (receiver) swaption with strike K , maturity T_α and tenor $T_\beta - T_\alpha$ (henceforth referred to also as $T_\beta \times (T_\beta - T_\alpha)$, or T_α -into- $(T_\beta - T_\alpha)$) is simply an option that gives the holder the right to enter at T_α into a payer (receiver) swap which matures at T_β and entitles to pay (receive) fixed rate K in exchange for floating LIBOR rate on the tenor dates \mathcal{T} . Thus, the payoff of the payer swaption with notional unit is given by

$$\max(S_{\alpha,\beta}(T_\alpha) - K, 0) \sum_{i=\alpha+1}^{\beta} \delta_i P(T_\alpha, T_i). \quad (\text{Eq. 2.9})$$

The expression $\sum_{i=\alpha+1}^{\beta} \delta_i P(T_\alpha, T_i)$ is sometimes called the annuity or present value per basis point (PVBP). Before the crisis it was a market practice to quote swaptions prices using a Black-like formula. Nowadays, to account for the all-too-real possibility of negative rates, market participants have shifted to using the so called Bachelier or normal model instead, in which the risk-neutral dynamics of the forward swap rate is normal rather than log-normal. In this approach, the time zero price of the above payer swaption is given by:

$$\begin{aligned} \text{PS}_{\alpha,\beta}(0, K) &= \\ &= \sum_{i=\alpha+1}^{\beta} \delta_i P(0, T_i) \left[(S_{\alpha,\beta}(0) - K) \Phi \left(\frac{S_{\alpha,\beta}(0) - K}{\sigma \sqrt{T_\alpha}} \right) + \varphi \left(\frac{S_{\alpha,\beta}(0) - K}{\sigma \sqrt{T_\alpha}} \right) \sigma \sqrt{T_\alpha} \right], \end{aligned} \quad (\text{Eq. 2.10})$$

where Φ and φ are the Gaussian cumulative and probability distribution functions respectively.

Finally, a Bermudan receiver (payer) swaption is an option to enter at any time T_i , $i \in \{\alpha, \alpha + 1, \dots, \beta - 1\}$, into a swap which terminates at T_β and gives the holder the right to receive (pay) a pre-determined fixed rate K in exchange for floating Libor. The period up to T_α is called the lockout or no-call period, and hence a Bermudan swaption with final exercise date $T_{\beta-1}$ and first exercise T_α is often called " T_β no-call T_α ," or " T_β nc T_α ." For instance, a 11nc1 swaption with annually

spaced exercise dates can be trained at the beginning of any year, starting from year 1. By exercising the option, the holder enters a swap starting at the time of exercise (i.e. years 1, 2, 3,..., 10) and ending at year 11⁶.

2.2. A loan contract

To fix ideas we focus in this paper on mortgage loans – i.e. loans taken for the purchase of a dwelling – since they tend to have relatively long maturities (ranging up to 30 years) that make the choice of a fixed vs. floating rate and the inherent prepayment optionality most acute. However, since we focus on the economics of the transaction rather than its legal characteristics, the ensuing discussion of the loan contract nature is purposefully somewhat vague and general. Broadly speaking, a mortgage loan is simply a contract whereby one party (“Client”) borrows a certain notional amount N at T_0 from another party (“Bank”) and commits to return it to the lender by T_N under the conditions stipulated in the contract. A loan contract will therefore specify i.a. the following features:

- ❖ applicable interest rate: this can be either a fixed rate K set at T_0 for the entire duration of the contract or a variable (floating) rate determined according to the prevailing market conditions which typically amounts to using the going 3M LIBOR (EURIBOR, WIBOR etc.) rate plus a spread compensating the bank for credit risk and potentially reflecting also other business considerations (competitive pressure etc.)⁷;
- ❖ amortization schedule: the capital can be either returned in a single payment at maturity – with only periodic interest cash flows in the interim – or repaid gradually at a predefined pace in equal or decreasing installments; to facilitate the presentation the focus is put below on the case of constant installment only, but the results carry over naturally also to other mortgage types;
- ❖ early termination conditions: whether and at what extra charge – if any – the outstanding loan can be paid back (or refinanced) prior to maturity, so called prepayment.

In a competitive market the pricing of a loan is determined in such a way that both the bank and the client are in principle indifferent between the fixed and floating rate mortgages with the same maturity, amortization schedule etc. This equivalence of the two rates ensures that no risk-less arbitrage is possible and the quoted fixed rate reflects the time-zero path of forward LIBOR rates. Thus, the net

⁶ Alternatively, such structure can be called a 1Y × 10Y, or one-into-ten, receiver, exercisable annually after the first exercise date.

⁷ Hybrid options are also possible whereby the interest rate is fixed for some initial part of the contract duration (e.g. 5 or 10 years) and floating thereafter. Since this specification does not present any additional technical difficulties, it is ignored below to ease the presentation.

present value of installments for a client paying on a floating rate basis – i.e. LIBOR plus spread – should be equal to the net present value of installments calculated according to a fixed rate K . Let us assume for the sake of the demonstration that the mortgage in question is non-amortizing (“interest rate only”) with no prepayment allowed. Then the remarks above can be formally restated as:

$$N \sum_{i=1}^M \delta_i ((L(T_0; T_{i-1}, T_i) + s) - K) P(T_0, T_i) = 0. \quad (\text{Eq. 2.11})$$

This in turn implies that the fixed rate K on an interest rate only mortgage loan is equal to the par forward swap rate plus spread and hence – given the spread s – can be calculated using the term structure of interest rates using:

$$K = \frac{\sum_{i=1}^M \delta_i L(T_0; T_{i-1}, T_i) P(T_0, T_i)}{\sum_{i=1}^M \delta_i P(T_0, T_i)} + s = \frac{1 - P(T_0, T_M)}{\sum_{i=1}^M \delta_i P(T_0, T_i)} + s. \quad (\text{Eq. 2.12})$$

In the more common case of a mortgage with amortizing capital $\{N_i\}_{i=1}^M$, (2.12) would feature instead a par forward swap rate for a contract with notional corresponding to the chosen amortization schedule⁸. Equation (2.12) makes clear that the interest rate risk inherent in a fixed rate mortgage without prepayment option can be perfectly offset using an interest rate swap with corresponding maturity and notional.

When clients are allowed to prepay their outstanding notional equation (2.12) should be adjusted by the spread component s_{opt} reflecting the fair value of the prepayment option:

$$K_{fixed} = K + s + s_{prepay}. \quad (\text{Eq. 2.13})$$

Note that since the prepayment option gives the client the right to “put” the loan principal to the bank, it is effectively a Bermudan receiver swaption, **RBS**, with first exercise date T_1 and swap termination date T_M . This involves a circular reference, since s_{prepay} depends on the value of the swaption and the value of the swaption in turn depends on the strike (fixed rate of the loan). The circularity can be overcome through the use of the following iterative procedure. Start by calculating **RBS**⁽⁰⁾ for the initial strike $K+s$. Since s_{prepay} represents the annuity-weighted value of the swaption, we have:

⁸ However, since this case does not alter anything in the substance of the argument but makes presentation less streamlined, it is omitted below.

$$s_{prepay}^{(1)} = \frac{\mathbf{RBS}^{(0)}}{\sum_{i=1}^M \delta_i P(T_0, T_i)}. \quad (\text{Eq. 2.14})$$

We can now re-price the swaption at a new strike, $K + s + s_{prepay}^{(1)}$, obtaining $\mathbf{RBS}^{(1)}$ which by analogy with (2.14) yields $s_{prepay}^{(2)}$. We continue in this fashion until the calibration stabilizes and the difference $s_{prepay}^{(n)} - s_{prepay}^{(n+1)}$ is, say, of the order of one basis point.

3. BREAKEVEN VOLATILITY

The concept of breakeven volatility was originally introduced in an unpublished note by Dupire (2006) who raised the problem of determining implied volatilities for options with different strikes and maturities given as sole information the historical price series of the underlying instrument⁹. Classical volatility estimation techniques typically yield a single number defined as the annualized standard deviation of log-returns:

$$\sigma_{hist} \equiv \sqrt{\frac{252}{N-1} \left(\sum_{i=1}^N \ln \left(\frac{S_i}{S_{i-1}} \right)^2 - \left(\frac{\ln(S_N/S_0)}{N} \right)^2 \right)}. \quad (\text{Eq. 3.1})$$

where S_i is the price of the underlying on day i . This procedure – inherently based on the assumption of constant volatility – would produce a single volatility parameter for all options on S . However, there is ample evidence that volatility is not in fact constant, and as a result the market participants tend to price options in such a way that different strike levels and maturities are associated with different implied volatility levels for the underlying – so called implied volatility “smile” or “skew” (Figure 1).

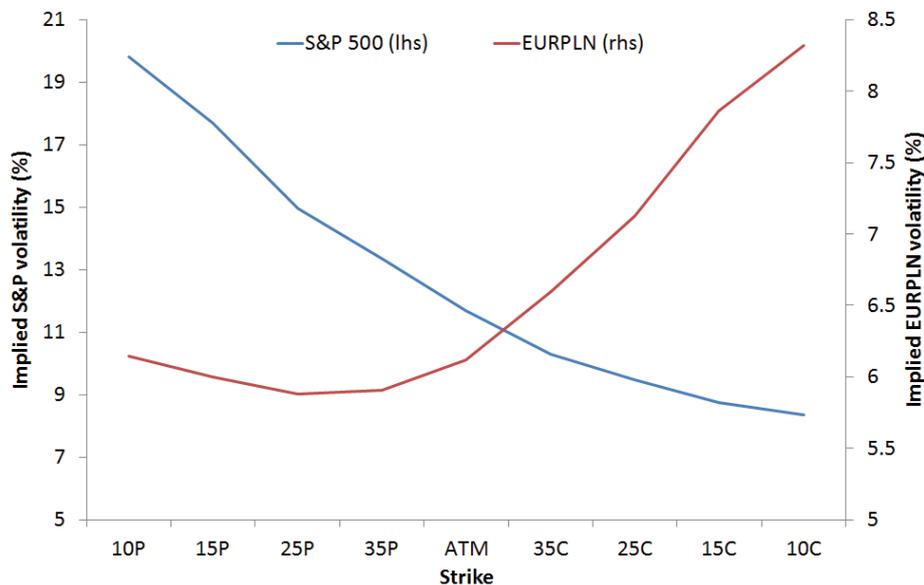
To account for this, Dupire¹⁰ suggests an approach based on back-testing of delta-hedged option strategies. The underlying idea bases on the recognition due to Black and Scholes¹¹ that dynamically hedging an option by removing its delta, i.e. first-order dependence on the price of the underlying instrument – the process referred to as “delta hedging” – transforms an initial premium into the final payoff

⁹ Thus, Dupire: writes: “Many people have devoted considerable time and effort to develop models that are calibrated to the market, usually in view of pricing exotic options. However, a possibly more fundamental question is: what the market should be?”

¹⁰ B. Dupire, *Pricing...*, *op. cit.*

¹¹ F. Black, M. Scholes, *The pricing of options and corporate liabilities*, *Journal of Political Economy* 1973, 81(3), 637–654.

Figure 1. Strike-dependent implied volatility pattern for S&P500 and EURPLN options with maturity 3M (as of 18 May 2017)

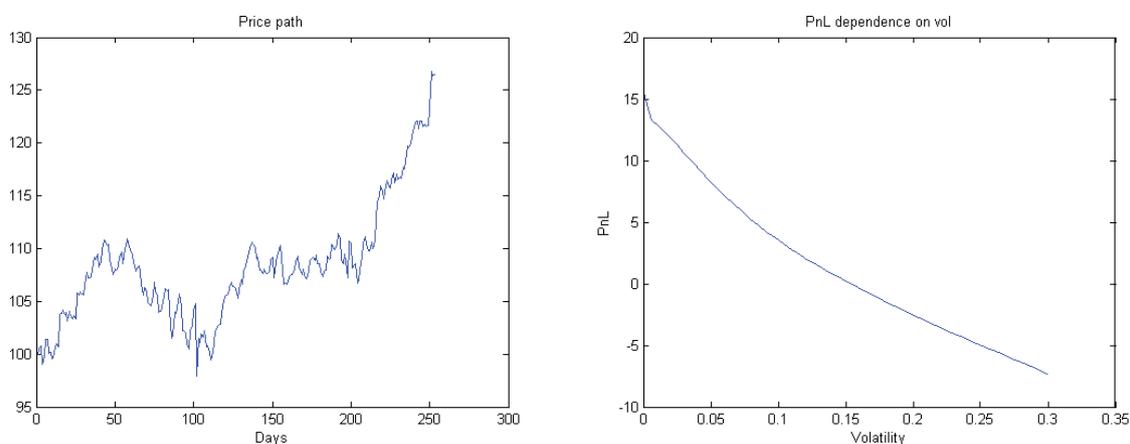


through replication. Thus, if one knows the volatility of a stock, one can replicate an option payoff exactly by continuously rebalancing a portfolio consisting of delta units of the underlying instrument and a risk-free bond. If no arbitrage is possible, then the value of the option should be equal to the cost of the replication strategy. In other words, given a path of the underlying instrument, hedging an option along this path using the model delta in principle allows to replicate the option. Leveraging this insight, if we sell an option for a premium corresponding to some volatility σ and then use the σ to calculate the option's delta along a path of the underlying instrument then by rebalancing the replication portfolio we finally end up with a profit or loss that depends on the volatility parameter σ . The value of σ that sets this profit and loss equal to zero is called the breakeven volatility. Figure 2 demonstrates this procedure for a stylized case of a call option on a generic asset S with strike price $K = 110$ and 1 year maturity. Here, breakeven volatility turns out to be 15.33%. Crucially, a different strike would lead to a different breakeven volatility. For instance, with a strike $K = 80$ instead, profit-canceling volatility would be just 3.5%. An alternative approach of producing a strike-dependent volatility pattern would consist in modeling the time series as a parametrized stochastic process then estimating the parameters to eventually price swaptions. A popular example is the Heston¹² (1993) model which features a classic Black-

¹² S.L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies 1993, 6(2), 327–343.

Scholes dynamics for the underlying instrument, but with a stochastic variance which follows a mean-reverting process of the type proposed by Cox, Ingersoll and Ross¹³. Atiya and Wall¹⁴ show how to obtain the maximum likelihood estimates of Heston model parameters (in the physical measure). However, the problem with such an approach is that parameter estimates are maturity-specific, so that option prices with different maturities are priced using different sets of parameters. Moreover, the model is significantly more numerically involved and drops the completeness inherent in the Black-Scholes framework by an introduced new stochastic driver for the volatility process, which complicates delta hedging.

Figure 2. The stylized path of the underlying instrument (the left hand panel) and the associated breakeven volatility (the right hand panel)



3.1. Swaption delta hedging

We now show how to adapt the Dupire's breakeven volatility concept to the case of interest rate swaptions. One might recall that the market quotes option prices using the Bachelier or normal model so that the fair value of a payer swaption is given by (2.10), i.e. expressed explicitly as the sum of the underlying swap and a portfolio of zero coupon bonds – the annuity. By analogy with the Black-Scholes approach, these two quantities become hedging instruments and the hedge ratios can be inferred directly from the equation. In particular, the hedge replicating the swaption

¹³ Heston model is in fact a continuous time analogue of models in the GARCH family. J.C. Cox, J.E. Ingersoll Jr, S.A. Ross, *A theory of the term structure of interest rates*, *Econometrica: Journal of the Econometric Society* 1985, pp. 385–407.

¹⁴ A.F. Atiya, S. Wall, *An analytic approximation of the likelihood function for the Heston model volatility estimation problem*, *Quantitative Finance* 2009, 9(3), 289–296.

(short position in a payer) consists in going long $\Delta = \Phi\left(\frac{S_{\alpha,\beta}(0)-K}{\sigma\sqrt{T_\alpha}}\right)$ units of the underlying forward swap contract and going short $\varphi\left(\frac{S_{\alpha,\beta}(0)-K}{\sigma\sqrt{T_\alpha}}\right)\sigma\sqrt{T_\alpha}$ units of the PVBP. The portfolio positions are then adjusted at discrete intervals as time goes by and the forward swap rate changes. Any net amount is invested/borrowed in the bond portfolio to ensure the portfolio is self-financing. If the replication was performed perfectly, with continuous re-hedging, the difference between the value of the hedging portfolio and terminal swaption payoff – i.e. the profit/loss, P&L – would be exactly zero, irrespective of the path taken by the swap rate. This observation justifies the statement that the Bachelier’s model provides a fair value of the swaption. Insofar as the replication strategy involves discrete rather than continuous rebalancing the P&L may deviate from zero but should be distributed symmetrically around it.

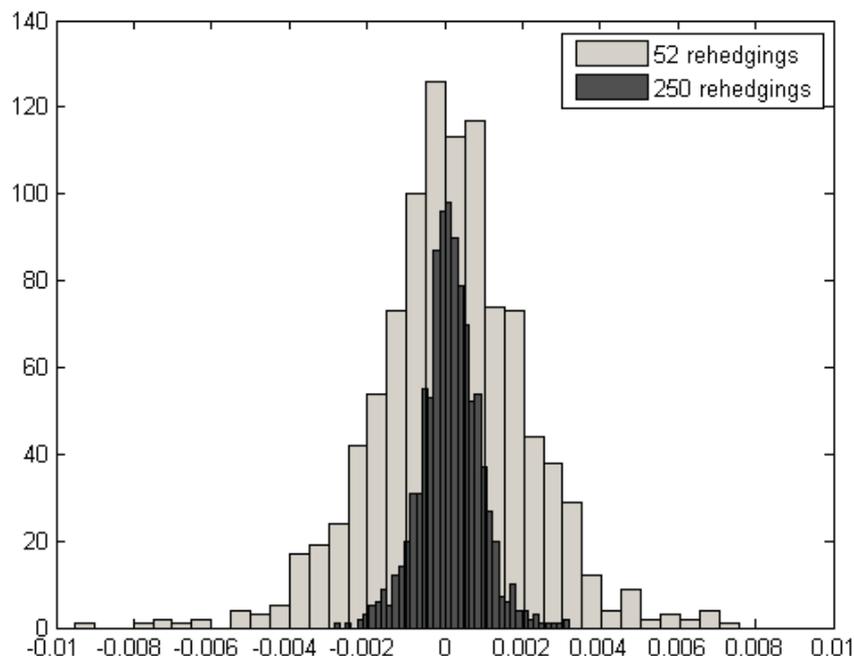
As an illustration considers a 1Y-into-5Y payer swaption in the Polish market struck at the money and sold at implied normal volatility of 70 bp.

We simulate the replication error using 10,000 paths for the underlying swap rate and use Polish interest rate curve data as of 30 December 2016. The simulation is carried out on a set of discrete equi-spaced times between time $t_0 = 0$ and swaption maturity, $T_1 = 1$. The hedging proceeds as follows:

- ❖ at $t_0 = 0$ short one unit of the 1x5 swaption, $\mathbf{PS}_{1,6}(0)$, long Δ_0 units of the underlying forward swap and short $\varphi(0)\sigma\sqrt{T_\alpha}$ units of the $PVBP_0$ so that the value of the portfolio (net cash flow from all transactions) is zero;
- ❖ at t_1 the underlying swap rate grows to $S_{1,6}(t_1)$ and swaption price changes to $\mathbf{PS}_{1,6}(t_1)$; thus we go long $\Delta_1 - \Delta_0$ units of the underlying forward swap and borrow/invest the resulting cash flow in the annuity bond portfolio whose value in the meantime has grown to $PVBP_1$;
- ❖ at each successive step until swaption expiration the hedge ratio is adjusted to keep the portfolio delta neutral and the resulting cash flows are invested/borrowed in the numeraire account.

Figure 3 shows the simulated PnL distribution in two cases – when the rebalancing is performed once per week (52 times per year) and daily (250 rehedgings). As expected, both distributions are centered around zero, but more frequent hedging produces visibly less dispersed the results.

Figure 3. Effect of delta hedging the 1x5 swaption: simulated profit and loss distributions (10,000 Monte Carlo runs)



3.2 Breakeven volatility for PLN swaptions

As of this writing there is no liquid market for swaptions involving Polish zloty (PLN). Hence, to come up with an assessment of what swaption prices could conceivably be, we can resort to the estimation of breakeven volatility surface using historical interest rate data. As explained above, in this approach the volatility at a given strike is chosen in a way to nullify the P&L accrued by daily delta-hedging of a swaption at that strike. Since our ultimate goal is to price a prepayment option in a mortgage contract we need estimates of volatilities for swap rates terminating at a common fixed date corresponding to the maturity of the mortgage which we set to 20 years¹⁵. Thus, we will estimate implied breakeven volatilities of the following 19 co-terminal swaptions: 1x19, 2x18, 3x17,...,19x1 as of January 2017. To mimic the convention in developed derivatives markets and provide a sufficiently broad set of calibration instruments, for each term/tenor we derive swaption implied volatilities for a range of strikes covering the par forward swap rate (the at-the-money, ATM contract) and $\text{ATM} \pm 200\text{bp}$, $\pm 100\text{bp}$, $\pm 50\text{bp}$ and $\pm 25\text{bp}$. For each term $T=1,2,\dots,19$ years we select a corresponding historical time point t such that $t+T$ is exactly the end of our data sample, i.e. 30 December 2016. We then calculate the

¹⁵ According to the Polish Bank Association (ZBP) data, roughly 64% of mortgages taken out in q4 2016 had contractual maturity between 25 and 35 years; 25% had maturity between 15 and 25 years and 11% – maturity below 15 years.

Bachelier's price of each swaption plugging a trial volatility σ into (2.10) and use historical interest rate data to calculate the P&L from delta hedging the swaption daily from origination at t until maturity on 30 December 2016. The breakeven volatility is then that choice of σ which sets the P&L from delta hedging, equals to zero and it is calculated numerically using a standard root finding algorithm. Since breakeven volatility is an estimate, it will generally depend on the time window chosen for the delta hedging. This calls for using averaged estimates over multiple non-overlapping historical time windows, which however is problematic given the long maturities of the swaptions considered. For instance, for the 19x1 swaption there is only one long enough time window. Yet, even if observations from the distant past were available, they would likely come from a different volatility regime so their practical relevance could be questionable. Moreover, even for shorter maturities for which historical data is available, running an iterative root-finding algorithm separately for each time window would be very costly in terms of computational time. Therefore, we decide against averaging breakeven volatilities, keeping in mind the approximate nature of the estimates. Figure 4 shows a sample breakeven volatility smile for the 1x19 swaption plotted against an actual implied volatility for 1x19 swaptions quoted in the most liquid US dollar market (sourced from Bloomberg as of 30 December 2016). Clearly, the pattern of the estimated breakeven volatilities is consistent with levels and shapes in more liquid markets. Figure 5 presents the entire estimated breakeven volatility surface for all term/tenor pairs. We may conclude that the surface exhibits plausible volatility levels and smile-like shapes and hence can serve as a basis for calibration.

Figure 4. Estimated breakeven volatility smile for the 1x19 swaption in Poland and actual implied volatility smile for the 1x19 USD swaption (as of 30 December 2017)

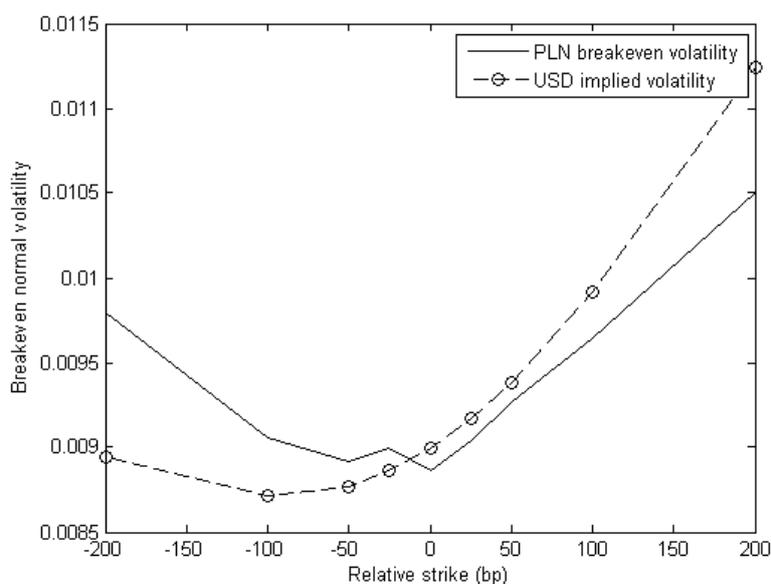
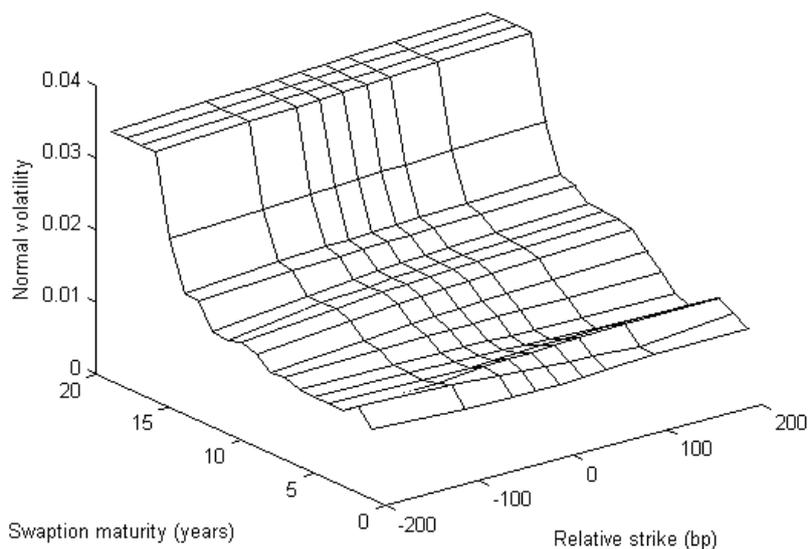


Figure 5. The estimated PLN normal breakeven volatility surface as of 1 January 2017 (fixed swap terminal is 20Y)



4. PRICING PREPAYMENT OPTION

As we have seen above, the estimated breakeven volatilities exhibit a consistent smile-like pattern across the maturity spectrum. This is clearly inconsistent with the Black-Scholes/Bachelier framework in which volatility is an inherent feature of the underlying instrument and should not exhibit dependence on strike. We overcome this problem by using a local, or state- and time-dependent, volatility version of the Cheyette model as suggested by Gatarek, Jablecki, and Qu¹⁶ and Gatarek and Jablecki¹⁷ whose reasoning we briefly summarize below adapting it to the case of co-terminal swaptions.

4.1 Cheyette local volatility model

Note that the introduction of non-parametric volatility in interest rate space is non-trivial. By convention, the fixing date of the swap coincides with the maturity of the option, i.e. swaptions with maturities T_α and $T_{\alpha+1}$ are written on two different underlyings evolving according to two different (forward) processes. As a result, unlike in traditional asset classes, options on swap rates are quoted only for one expiry and swaption prices cannot be differentiated with respect to

¹⁶ D. Gatarek, J. Jablecki, D. Qu, *Non-parametric local volatility formula for interest rate swaptions*, Risk 2016, pp. 120–124.

¹⁷ D. Gatarek, J. Jablecki, *A local volatility model...*, *op. cit.*

expiration time. It is thus *prima facie* impossible to analyze the time evolution of swaption implied distribution functions and recover from them – via the Forward-Planck equation – the unique swap rate diffusion generating them, as originally proposed for equities by Dupire¹⁸. We circumvent this problem by introducing a fixed-tenor rolling maturity swap rate and deriving a spot process for it.

Let $0 < T_\alpha < T_\beta$ be two maturities and consider the forward swap rate with fixing date T_α and maturity T_β as defined in (2.8) (from here on, without loss of generality we shall, for simplicity, use continuous-time rather than discrete convention). The forward swap rate is, by definition, a martingale under the measure $\mathbb{Q}^{\alpha,\beta}$ associated to the annuity numeraire $N_{\alpha,\beta}(t) \equiv \int_{T_\alpha}^{T_\beta} P(t, s) ds$, i.e. $S_{\alpha,\beta}(t)$ has the driftless dynamics under $\mathbb{Q}^{\alpha,\beta}$:

$$dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}(t) dW^{\alpha,\beta}(t), \quad (\text{Eq. 4.1})$$

where $\sigma_{\alpha,\beta}$ is a continuous stochastic process and $W^{\alpha,\beta}(t)$ is a Brownian motion under $\mathbb{Q}^{\alpha,\beta}$.

For a given swap maturity date T , we define the fixed-terminal rolling swap rate as

$$S_T(t) \equiv S_{t,T}(t) = \frac{1 - P(t, T)}{\int_t^T P(t, s) ds} \quad (\text{Eq. 4.2})$$

Note that $S_T(t)$ is a spot instrument, albeit not a traded one, and it is not a martingale. However, using (4.1), its dynamics can be derived to be:

$$dS_T(t) = Q_{t,T}(S_T, t) dt + \sigma_{t,T}(t) dW^{t,T}(t) \quad (\text{Eq. 4.3})$$

where $Q_{t,T}(S_T, t) \equiv \left. \frac{\partial S^{u,T}(t)}{\partial u} \right|_{u=t}$ and $W_{t,T}(u)$ is a Brownian motion under the measure $\mathbb{Q}^{t,T}$ defined as

$$dW_{t,T}(u) = dW(u) + \frac{\int_t^T B(u, s) \Sigma(u, s) ds}{\int_t^T B(u, s) ds} du. \quad (\text{Eq. 4.4})$$

Having done some algebra, $Q_{t,T}(S_T, t)$ can be represented as

$$Q_{t,T}(S_T, t) = S_T(t) \left[\frac{S_T(t) - r(t)}{1 - P(t, T)} \right]. \quad (\text{Eq. 4.5})$$

¹⁸ B. Dupire, *Pricing...*, *op. cit.*

Let us assume now that the forward swap rate volatility is a deterministic function of the swap rate and time, $\sigma_{t,T}(t) \equiv \sigma_{t,T}(t, S_T)$. It can be shown that $\sigma_{t,T}(t, S_T)$ is given in terms of swaptions prices by the following Dupire-type equation (since the common swap maturity T is fixed, swaption dependence on T is suppressed):

$$\sigma_{t,T}(t, K) = \sqrt{\frac{\partial_t C(t, K) + \partial_K C(t, K) (Q_{t,T}(S_T, t) + q(t, T))}{\frac{1}{2} \partial_K^2 C(t, K)}}, \quad (\text{Eq. 4.6})$$

where $q(t, T)$ is an adjustment due to the differentiation of swaption prices with respect to maturity. Since $q(t, T)$ has been found to be very small (Gatarek and Jablecki¹⁹; Qu²⁰), in practice the local volatility function can be approximated by:

$$\sigma_{t,T}(t, K) \approx \sqrt{\frac{\partial_t C(t, K) + \partial_K C(t, K) Q_{t,T}(S_T, t)}{\frac{1}{2} \partial_K^2 C(t, K)}}. \quad (\text{Eq. 4.7})$$

Through straightforward differentiation of the undiscounted Bachelier swaption formula $C = (F_T - K) \Phi((F_T - K)/\sigma/\sqrt{T}) + \varphi((F_T - K)/\sigma/\sqrt{T}) \sigma \sqrt{T}$, equation (4.7) can also be recast in terms of normal implied volatilities Σ ²¹:

$$\sigma_{t,T}(t, K) = \sqrt{\frac{2 \frac{\partial \Sigma}{\partial t} + \frac{\sigma}{t} + 2Q(t, T) \frac{\partial \sigma}{\partial K}}{\frac{1}{\sigma t} \left(1 + \frac{(F_T(t) - K)}{\sigma} \frac{\partial \sigma}{\partial K}\right)^2 + \frac{\partial^2 \sigma}{\partial K^2}}}, \quad (\text{Eq. 4.8})$$

where $F_T(t) = S_{t,T}(0) \exp(\int_0^t Q(s, T) ds)$ is the forward rolling swap rate and $\Phi(\cdot)$, $\varphi(\cdot)$ are standard normal CDF and PDF respectively. Plugging (4.8) into (4.3) yields local volatility diffusion for the rolling swap rate.

Pricing interest rate derivatives in general requires not only the simulation of swap rate paths, but a fully-fledged interest rate model calibrated to the time zero interest rate curve. Fortunately, swap rate local volatility (4.8) can be easily virtually fed into any generic model, such as e.g. Libor Market Model or Cheyette²²

¹⁹ D. Gatarek, J. Jablecki, *A local volatility model...*, *op. cit.*

²⁰ D. Qu, *Manufacturing and managing customer-driven derivatives*, John Wiley & Sons, Chichester 2016, West Sussex, United Kingdom.

²¹ Strictly speaking, these volatilities will be associated with the rolling swap dynamics (4.3), whereas implied volatilities quoted by the market are those of the forward swap process (4.1). Fortunately, under the approximation $q(t, T) = 0$, the two volatility parameters coincide.

²² O. Cheyette, *Term structure dynamics and mortgage valuation*, *The Journal of Fixed Income* 1992, 1(4), 28–41.

model. The latter is a particularly convenient choice as it admits a two-dimensional Markovian representation of the entire yield curve dynamics. Specifically, the Cheyette model is given by:

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left(-\frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)} \right) x(t) - \frac{1}{2\kappa^2} \left(1 - e^{-\kappa(T-t)} \right)^2 y(t) \right), \quad (\text{Eq. 4.9})$$

where $x(t)$ and $y(t)$ are state variables and κ is a constant positive number representing mean reversion speed. The mean reversion speed. The two state variables have the following dynamics:

$$dx(t) = (y(t) + -\kappa x(t)) dt + \sigma(t) dW(t) \quad (\text{Eq. 4.10})$$

$$dy(t) = (\sigma^2(t) - 2\kappa y(t)) dt \quad (\text{Eq. 4.11})$$

State variable $x(t)$ has the interpretation of a centered short rate, while $y(t)$ is an upward drift representing forward curve steepening due to volatility (a “convexity correction”). Since rolling swap rates are a function of bond prices, straightforward application of Ito’s lemma reveals that the volatilities of the swap rate and the short rate in the Cheyette model are linked through:

$$\sigma(t) = (\partial_x S(t, x(t), y(t)))^{-1} \sigma^{t,T}(t). \quad (\text{Eq. 4.12})$$

With swap rate local volatility stripped from the breakeven volatility surface (Figure 5) via (4.8) and then mapped to the short rate volatility through (4.12), Cheyette model can be implemented in a standard Monte Carlo pricer. The procedure mimics closely the well-established routine of calibrating local volatility models in equity or FX space (cf. Gatarek and Jablecki²³ for details; see also Qu²⁴ for a PDE implementation).

4.2. Bermudan swaption pricing

As discussed/ presented above, the prepayment option contained in a fixed rate mortgage is of Bermudan character. The author explained above how it can be handled in a Monte Carlo setting. Consider a “ T_β no-call T_α ” Bermudan receiver swaption introduced above. The time t value of such a Bermudan swaption will be denoted $\mathbf{RBS}_{\alpha,\beta}(t, K)$. Assuming no prior exercise, at any time point T_n , the

²³ D. Gatarek, J. Jablecki, *A local volatility model...*, *op. cit.*

²⁴ D. Qu, *Manufacturing and managing...*, *op. cit.*

swaption holder has the right to receive the exercise value V_e of the swaption, i.e. present value of the underlying swap:

$$V_e(T_n) \equiv (K - S_{n,\beta}(T_n))^+ \sum_{k=n+1}^{\beta} P(T_n, T_k) \delta_k. \quad (\text{Eq. 4.13})$$

The exercise value has to be compared to the so called continuation value, V_c , of holding the option beyond T_n :

$$V_c(T_n) \equiv \mathbb{E} \left(\mathbf{RBS}_{\alpha,\beta}(T_{n+1}, K) \middle| S_{n,\beta}(T_n) \right). \quad (\text{Eq. 4.14})$$

The value of the Bermudan swaption can now be given in terms of (4.13) and (4.14) via a dynamic programming recursion:

$$\begin{aligned} \mathbf{RBS}_{\alpha,\beta}(T_{\beta-1}, K) &= P(T_{\beta-1}, T_{\beta}) \delta_{\beta} (K - S_{\beta-1,\beta}(T_{\beta-1}))^+ \\ \mathbf{RBS}_{\alpha,\beta}(T_j, K) &= \max(V_e(T_j), V_c(T_j)) \end{aligned} \quad (\text{Eq. 4.15})$$

for $j = \beta - 2, \beta - 3, \dots, n$. The evaluation of (4.15) proceeds backward in time: at $T_{\beta-1}$ the value of the Bermudan swaption is known and determined by the standard swaption payoff. This allows us to update the continuation value at $T_{\beta-2}$ by discounting and compare it to the exercise value prevailing at the time. The procedure of comparing “backwardly-cumulated” continuation value with the immediate exercise value and deciding upon swaption exercise is repeated until the initial valuation date is reached, at which point the algorithm yields a price estimate for the Bermudan swaption. Handling such a problem in a Monte Carlo setting can be challenging. The idea going back to Longstaff and Schwartz²⁵ is that the continuation value at each time step can be approximated by its least-squares conditional forecast, \widehat{V}_c , thus allowing us to resolve the decision rule (4.15) without “seeing into the future.” Specifically, the continuation value is represented as a linear combination of M basis functions $\psi(\cdot)$ (see Brigo and Mercurio²⁶ for an excellent general discussion of the method):

$$V_c(T_n) \approx \widehat{V}_c(T_n) \equiv \sum_{j=1}^M \lambda_{nj} \psi_j(T_n), \quad (\text{Eq. 4.16})$$

²⁵ F.A. Longstaff, E.S. Schwartz, *Valuing American options by simulation: a simple least-squares approach*, Review of Financial studies 2001, 14(1), 113–147.

²⁶ D. Brigo, F. Mercurio, *Interest rate models-theory and practice: with smile, inflation and credit*, Springer Science & Business Media 2007.

with weights λ_j determined by least-squares regression. This requires first simulating a sufficient number N of yield curve scenarios which produces a set of swap rates $(S_{\alpha,\beta}(T_n), S_{\alpha+1,\beta}(T_n), \dots, S_{\beta-1,\beta}(T_n))_k$, $n = \alpha, \dots, \beta$, $k = 1, \dots, N$. With “perfect foresight” knowledge of each simulated path k , the exercise and continuation values, as well as Bermudan swaption prices $\mathbf{RBS}_{\alpha,\beta}(T_i, K)$, can be evaluated recursively along each path using (4.15). To improve quality of fit and run-time performance only in-the-money paths are considered for the estimation of the weights λ_j . With the estimated regression coefficients, the same Monte Carlo swap rate paths are then used to determine the approximate continuation values $\widehat{V}_c(T_n)$ and Bermudan swaption payoffs for each path. It should be stressed, however, that this produces a lower bound estimate of the Bermudan swaption price.

4.3. Numerical example

To demonstrate the viability of our method we price a prepayment option contained in a 20-year Polish zloty mortgage issued at the beginning of 2017. To facilitate presentation (but with no substantial loss in generality) let us assume the mortgage has a simple interest-only (no amortization) structure and a principal of PLN 250,000, which roughly corresponds to the average value of mortgage loans taken out in Poland in 2016. We also assume interest is payable on a semi-annual basis. Using interest rate curve data as of 30 December 2016 we find that the 0.5y-into-19.5y swap rate equals 0.036. From Section 2.2 one can recall that a fixed rate mortgage rate can be decomposed into the swap rate with corresponding maturity, credit spread (which we take to include also other, non-credit business considerations) and the spread compensating the bank for the prepayment risk inherent in the fixed rate mortgage. Credit spread levels can be inferred from the NBP’s database on the new sale of floating rate loans as the difference between the quoted interest rate on a floating rate mortgage and the WIBOR 3M rate. Although credit exposure in a fixed rate mortgage could differ from that in a floating rate, the floating rate spread is actually the best readily available approximation to the potential spread in a fixed rate mortgage. The spread as of January 2017 stood at 280 bp, so that the fixed rate in a mortgage without prepayment option would be 6.40%.

To estimate the prepayment spread – and thus price in the prepayment option – we follow the iterative approach laid out in Section 2.2. This entails pricing a 20-year Bermudan swaption, to which end we employ the Cheyette local volatility model calibrated to the breakeven volatility surface and simulate optimal stopping time as explained in Section 4.2. The LSMC algorithm leads to fairly quick and stable convergence, so that using 10,000 paths with a time step of 1/12 seems satisfactory. The spread is found numerically to equal 184 bp and the total value of the prepayment option is PLN 155,475 – or over 60% of the mortgage capital.

This puts the fixed rate at 8.24% vs. 4.50% on floating rate mortgages in January 2017. The breakdown of the calculations is shown in Table 1.

Table 1. The simulated costs of a fixed rate mortgage (interest rate curve data for the Polish and USD market as of 30 December 2016)

	Swap rate	Credit spread	FRM (no prep.)	Prep. spread	FRM	Prep. option price
(share of notional)						
Poland (BEV)	3.60%	2.80%	6.40%	1.84%	8.24%	62.19%
USA	2.66%	1.00%	3.66%	1.38%	5.04%	46.84%

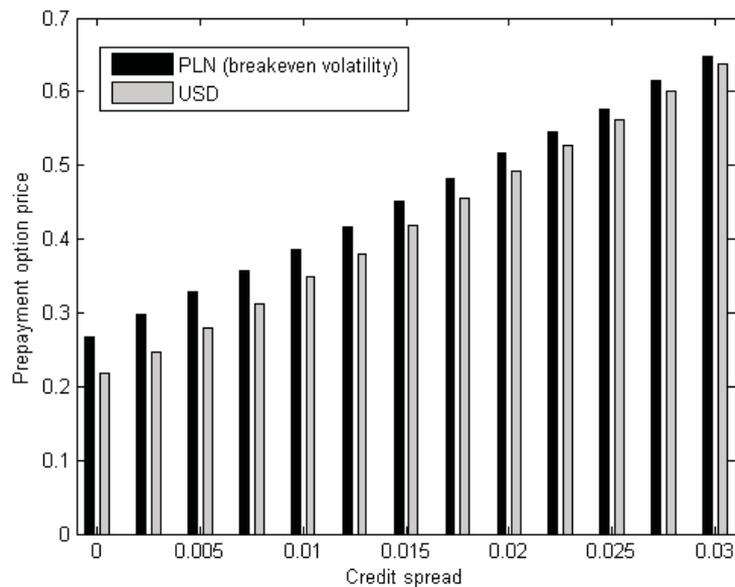
Note: FRM (no prep.) is the fixed rate mortgage rate if prepayment is not allowed; Prep. spread is the prepayment spread and FRM denotes the fixed rate mortgage rate accounting for the possibility of prepayment.

To provide a very basic robustness check of the results, the author compares compare the numbers found for Poland based on a breakeven volatility surface (Figure 5) to an analogous estimate for the United States based on swaption implied volatilities quoted in the USD market as of January 2017. While mortgage credit spreads for the US are not directly available, some indication as to their level may be provided by the margin for 5/1-year adjustable rate mortgage quoted by Freddie Mac. This mortgage offers a fixed rate for an initial period of 5 years and then resets to an index plus margin fixed once per year. ARM mortgage spread stood at about 270 bp in January 2017 but at least part of that reflects prepayment spread. For benchmarking purposes we thus set the credit spread in our 20y fixed rate mortgage at 100 bp. This results in a prepayment spread of 138 bp and a mortgage rate of 5.04%. This is close to the actual levels of the fixed rates on mortgages in the US market (e.g. the 30-year mortgage rate stood at 4.30% in January 2017, which given that the swap curve is virtually flat between the 20y–30y tenors should be roughly similar to the cost of a 20y mortgage for which unfortunately no national averages are reported).

Finally, to see how sensitive the total cost of prepayment option is to the assumed credit spread, we reprice the Bermudan option implicit in the respective contracts assuming credit spread levels in the range 0–300 bp (Figure 6). The results indicate that even if banks were to charge no credit spread or other margins, the cost of the prepayment option would still be substantial – about 25% and 20% of the mortgage notional for Poland and the US respectively. This suggests

that, especially in Poland where interest rates have historically exhibited relatively high volatility, the mortgage spread component related to the prepayment option tends to be quite significant, which underscores the importance of an adequate risk management of the inherent callability feature as indeed suggested by regulators.

Figure 6. The price of a prepayment option (a Bermudan receiver) in a 20Y mortgage as a function of loan credit spread



5. CONCLUSIONS

The goal of this paper was to suggest a methodology for estimating the value of a prepayment option in cases where a deep and liquid market in interest rate swaptions is not available. In such circumstances, it is a priori not clear how to calibrate the prepayment option pricing model, which compounds valuation uncertainty and possibly hinders the development of fixed rate mortgages. The proposed approach consists in adapting the concept of breakeven volatility to interest rate swaptions. In particular, to estimate what the unknown swaption volatilities could be, we suggested back-testing a delta hedged position in a theoretical swaption to find numerically the volatility level, which nullifies any accumulated hedging profit/loss. Since at that volatility the hedger breaks even, its level can be considered “fair” and serves as a basis for calibration. The proposed method has two main uses.

Firstly, it can be used to offer guidance on the likely cost of a fixed rate mortgage in markets where no such products have developed so far. This paper looks at the specific example of Poland, where the “only game in town” is a floating rate

mortgage. Specifically, having produced such a breakeven volatility surface for the Polish zloty interest rate market, we have employed the calibrated robust Bermudan swaption pricing model to estimate the fair value of prepayment spread in a stylized 20-year fixed rate mortgage. The prepayment spread component proves to be quite significant, stressing the importance of an adequate risk management of the inherent callability feature and possibly explains why fixed rate mortgage products have struggled to develop in Poland so far.

Secondly, our method can also be used in developed mortgage markets, where fixed rate contracts are available, to benchmark or assess the degree of potential mispricing in mortgage contracts, driven by the prepayment option.

Abstract

This paper presents a novel approach of estimating the value of a prepayment option in a fixed rate loan based on the concept of breakeven volatility. Since the prepayment option can be exercised essentially at any time prior to maturity, its valuing requires: (i) a pricing model sophisticated enough to handle its early exercise feature; and (ii) a broad set of interest rate derivatives prices to which the model can be calibrated to preclude arbitrage. This paper shows that when the derivatives market is not developed enough to ensure calibration, a good approximation of the fair value of a prepayment option can be derived by constructing the “missing” derivatives prices by back-testing delta hedged swaptions. This produces a “fair” volatility surface conditioned on the realized historical zero coupon bond prices and swap rates, which can be used to calibrate the prepayment option pricing model. The paper presents numerical examples for the Polish market as of January 2017. The mortgage spread component related to the prepayment option price proves to be quite significant, stressing the importance of an adequate risk management of the inherent callability feature and possibly explains why fixed rate mortgage products have struggled to develop in Poland so far.

Key words: prepayment, fixed rate loan, Bermudan swaption, breakeven volatility, Cheyette model

References

- Agarwal S., Driscoll J.C., Laibson D.I., *Optimal Mortgage Refinancing: A Closed-Form Solution*, Journal of Money, Credit and Banking 2013, 45(4), pp. 591–622.
- Atiya A.F., Wall S., *An analytic approximation of the likelihood function for the Heston model volatility estimation problem*, Quantitative Finance 2009, 9(3), pp. 289–296.

- Black F., Scholes M., *The pricing of options and corporate liabilities*, Journal of Political Economy 1973, 81(3), pp. 637–654.
- Brigo D., Mercurio F., *Interest rate models-theory and practice: with smile, inflation and credit*, Springer Science & Business Media 2007.
- Cheyette O., *Term structure dynamics and mortgage valuation*, The Journal of Fixed Income 1992, 1(4), pp. 28–41.
- Collin-Dufresne P., Harding J.P., *A closed form formula for valuing mortgages*, The Journal of Real Estate Finance and Economics 1999, 19(2), pp. 133–146.
- Cox J.C., Ingersoll Jr J.E., Ross S.A., *A theory of the term structure of interest rates*, Econometrica: Journal of the Econometric Society 1985, pp. 385–407.
- Davidson A., Levin A., *Mortgage Valuation Models: Embedded Options, Risk, and Uncertainty*, Oxford University Press 2014.
- Dupire B., *Pricing with a smile*, Risk 1994, 7(1), pp. 18–20.
- Dupire B., *Fair Skew: Break-Even Volatility Surface*, Discussion paper, Bloomberg 2006.
- Gatarek D., Jablecki J., *A local volatility model for swaptions smile*, Journal of Computational Finance 2016, Forthcoming.
- Gatarek D., Jablecki J., Qu D., *Non-parametric local volatility formula for interest rate swaptions*, Risk 2016, pp. 120–124.
- Heston S.L., *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies 1993, 6(2), pp. 327–343.
- Kau J.B., Keenan D.C., *An overview of the option-theoretic pricing of mortgages*, Journal of Housing Research 1995, 6(2), p. 217.
- Longstaff F.A., Schwartz E.S., *Valuing American options by simulation: a simple least-squares approach*, Review of Financial studies 2001, 14(1), pp. 113–147.
- Qu D., *Manufacturing and managing customer-driven derivatives*, John Wiley & Sons, Chichester 2016, West Sussex, United Kingdom.